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Advection-Dominated Accretion Flows with Causal Viscosity

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ABSTRACT

Aims. We present the basic equations and sample solutions for the steady-state global transonic solutions of the advection-dominated accretion flows (ADAFs) with causal viscosity prescription. The procedures of the stable numerical calculations and all explicit formula to obtain the solutions of ADAFs are also presented.

Methods. We solve the transonic solutions of ADAFs by using the explicit numerical integrations, such as the Runge-Kutta method. In this calculation method, we firstly solve the physical values at the sonic radius where L'Hopital's rule is used. Then, we numerically solve the coupled differential equations of the radial velocity, the angular momentum and the sound speed from the sonic radius to the inward and outward directions.

Results. By the calculation procedures presented in this paper, we can cover the all parameter spaces of the transonic solutions of ADAFs. Sample transonic solutions for ADAF-thick disk and ADAF-thin disk are presented. The explicit formula for the analytical expansion around the singular points, the sonic point and the viscous point, are presented. If we set the diffusion timescale to be null, the formalism in this study become the formalism of the acausal viscosity which is usually used in the past study for the calculation of the ADAF structure.

Key words. accretion: accretion disks—black hole physics—hydrodynamics

1. Introduction

Advection dominated accretion flows (ADAFs) are one of the fundamental solutions of the accretion flows into the black holes in the low-luminous systems, such as, low-luminosity active galactic nuclei. In past studies, the several methods are used to obtain the transonic solutions of ADAFs, e.g., the relaxation scheme (e.g. Narayan, Kato & Honma 1997), the implicit integration scheme (e.g. Kato, Fukue & Mineshige 1998), the outward integration scheme with the inner boundary condition (Becker & Le 2003) or the inward or outward integration scheme with the special condition at and inside the sonic radius (Lu, Gu & Yuan 1999). All these works assumes the diffusive viscosity with infinite diffusive velocity.

In this paper, we give the method to solve the global transonic solutions of ADAFs with causal viscosity prescriptions which assume the finite diffusive velocity proposed by Papaloizou

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& Szuszkiewicz (1994), which can become the alternative method to solve the global solutions. Gammie & Popham (1998) calculate the ADAF structure in the Kerr metric with the causal viscousity prescription by using the relaxation scheme. In the calculations with the causal viscosity prescription, the boundary condition at the viscous point is used, and the boundary condition at the horizon is not required (see, also §5). If we set the diffusion timescale to be null, the basic equations in this study become the basic equations for the case of the acausal viscosity which is usually used in the past study.

After we present the basic equations for the ADAF with the causal viscosity prescription in §2, the boundary conditions for the global transonic solutions of ADAFs with the causal viscosity are given in §3. The numerical procedures of the method to solve the transonic solutions of ADAFs and the sample solutions are presented in §4. Discussion and conclusions are given in §5 and §6, respectively.

2. Basic Equations of ADAF with Causal Viscosity

The basic equations used in this paper is same as those in Narayan, Kato & Honma (1997) except that we use the causal viscosity prescription. The mass conservation is written as,

$$-4\pi r H \rho v_r = \dot{M}, \tag{1}$$

where H is the half thickness of the accretion disk, ρ is the gas density, v_r is the radial velocity and $\dot{M}$ is the mass accretion rate. Here, we assume $v_r < 0$ and $H = (5/2)^{1/2} a_s/\Omega_K$ where a_s is the sound speed defined as $a_s = (p/\rho)^{1/2}$ and p is the pressure.

The radial momentum conservation is described as,

$$v_r \frac{dv_r}{dr} = (\Omega^2 - \Omega_K^2) r - \frac{1}{\rho}\frac{dp}{dr}, \tag{2}$$

where Ω is the angular velocity of the accretion flow and Ω_K is the Keplerian angular velocity calculated as $\Omega_K = 1/[r^{1/2}(r-2)]$. Here, we set $GM = 1$ and use the Paczyńskii-Wiita potential.

The energy equation is written as,

$$\frac{\rho v_r}{\gamma - 1}\frac{da_s^2}{dr} - a_s^2 v_r \frac{d\rho}{dr} = \rho v_r (\ell - j)\frac{d\Omega}{dr}, \tag{3}$$

where γ is the ratio of specific heats of the accreting gas, $\ell (= \Omega r^2)$ is the angular momentum, and j is the integration constant meaning the specific angular momentum at some radius.

For the case of the diffusive viscosity with the infinite diffusion velocity, the shear stress $t_{r\phi}$ is written as

$$t_{r\phi} = -\nu \rho r^2 \frac{d\Omega}{dr}. \tag{4}$$

Then, the angular momentum equation for the case of the acausal viscosity is written as,

$$\frac{d\Omega}{dr} = \frac{v_r(\ell - j)}{\nu r^2}, \tag{5}$$

where ν is the kinematic coefficient of viscosity, and, here, j is the specific angular momentum per unit mass where $d\Omega/dr = 0$, if such point exists. By assuming the α-viscosity we set $\nu = \alpha a_s^2/\Omega_K$. From Eq. (5), the equation for the angular momentum is written as

$$\frac{d\ell}{dr} = \frac{2\ell}{r} + \frac{v_r(\ell - j)}{\nu}. \tag{6}$$

In the same way as Papaloizou & Szuszkiewicz (1994) and Gammie & Popham (1998), we include the effects of the diffusive viscosity with the finite diffusion velocity. By using the causal viscosity, we can derive the angular momentum equation with the diffusive viscosity with the diffusion velocity a_ν written as,

$$\frac{d\ell}{dr} = \frac{\mathcal{N}_\nu}{\mathcal{D}_\nu}, \tag{7}$$

where

$$\mathcal{D}_\nu = 1 - v_r^2/a_\nu^2, \quad \mathcal{N}_\nu = \frac{2\ell}{r} + \frac{v_r(\ell - j)}{\nu}\left(1 - \frac{2v_r\tau_\nu}{r}\right), \tag{8}$$

Here, τ_ν is the relaxation timescale. For simplicity, we set τ_ν and a_ν as $\tau_\nu = \Omega_K^{-1}$ and $a_\nu = (\nu/\tau_\nu)^{1/2}$, respectively. Now, the shear stress for the causal viscosity is written as

$$t_{r\phi} = \frac{-\nu\rho}{1 - 2v_r\tau_\nu/r}\left[\left(1 - \frac{v_r^2}{a_\nu^2}\right)r^2\frac{d\Omega}{dr} - \frac{2\ell}{r}\frac{v_r^2}{a_\nu^2}\right]. \tag{9}$$

When $\tau_\nu = 0$, i.e., $a_\nu = \infty$, Eq. (7) and (9) for the causal viscosity become the equations for the acausal viscosity, Eq. (6) and (4). Even in the point where $d\Omega/dr = 0$, the shear stress $t_{r\phi}$ in general do not become null due to effects of the causal viscosity. Since, the effects of the causal viscosity are coupled with the radial velocity as v_r^2/a_ν^2 and $v_r\tau_\nu$ in Eqs. (7) and (9), the effects of the causal viscosity is effective in the inner region of the accretion flow where the absolute value of the radial velocity is large.

3. Boundary Conditions and Physical Values at the Sonic Radius

Since the accretion flows plunge into the black hole supersonically after a sonic transition, the equations are singular at the sonic radius. In order to see this, from Eqs. (1), (2) and (3), we can derive the equation for dv_r/dr as,

$$\frac{dv_r}{dr} = \frac{\mathcal{N}}{\mathcal{D}}, \tag{10}$$

where

$$\begin{aligned}
\mathcal{D} &= v_r - \left(\frac{2\gamma}{\gamma+1}\right)\frac{a_s^2}{v_r}, \\
\mathcal{N} &= \frac{\Omega^2 - \Omega_K^2}{r} - \left(\frac{2\gamma}{\gamma+1}\right)a_s^2\frac{d\ln(\Omega_K/r)}{dr} \\
&\quad - \left(\frac{\gamma-1}{\gamma+1}\right)\frac{(\ell - j)}{r^2}\left(\frac{d\ell}{dr} - \frac{2\ell}{r}\right).
\end{aligned} \tag{11}$$

Here, Eq. (7) is inserted to $d\ell/dr$ and $d\ln(\Omega_K/r)/dr = -(5r-6)/[2r(r-2)]$. On the other hand, the equation for da_s/dr is written as,

$$\frac{da_s}{dr} = \left(\frac{\gamma-1}{\gamma+1}\right)\left[-\frac{a_s}{v_r}\frac{dv_r}{dr} + a_s\frac{d\ln(\Omega_K/r)}{dr} + \frac{(\ell - j)}{a_s r^2}\left(\frac{d\ell}{dr} - \frac{2\ell}{r}\right)\right]. \tag{12}$$

In order to pass the sonic radius, r_s, smoothly, two boundary conditions at $r = r_s$ are given as $\mathcal{D} = \mathcal{N} = 0$ at $r = r_s$. For given $a_{s,s}$ (> 0) which is the sound speed at the sonic radius, the radial velocity $v_{r,s}$ at $r = r_s$ is calculated from $\mathcal{D} = 0$ as $v_{r,s} = -a_{s,s}[2\gamma/(\gamma+1)]^{1/2}$. From the condition

$\mathcal{N}=0$, the angular momentum ℓ_s at $r=r_s$ is calculated as, $\ell_s=[b_\ell \pm (b_\ell^2-a_\ell c_\ell)^{1/2}]/a_\ell$ where

$$a_\ell = 1-\left(\frac{\gamma-1}{\gamma+1}\right)\left\{-2+\frac{1}{\mathcal{D}_\nu}\left[2+\frac{rv_r}{\nu}\left(1-\frac{2v_r\tau_r}{r}\right)\right]\right\}, \tag{13}$$

$$b_\ell = \frac{j}{2}\left(\frac{1-\gamma}{1+\gamma}\right)\left\{-\frac{rv_r}{\nu\mathcal{D}_\nu}\left(1-\frac{2v_r\tau_r}{r}\right)\right. \\ \left. -2+\frac{1}{\mathcal{D}_\nu}\left[2+\frac{rv_r}{\nu}\left(1-\frac{2v_r\tau_r}{r}\right)\right]\right\}, \tag{14}$$

$$c_\ell = -\left(\frac{\gamma-1}{\gamma+1}\right)\frac{rv_r j^2}{\nu\mathcal{D}_\nu}\left(1-\frac{2v_r\tau_r}{r}\right)-\Omega_K^2 r^4 \\ -\frac{2\gamma}{\gamma+1}a_s^2 r^3\frac{d\ln(\Omega_K/r)}{dr}. \tag{15}$$

Chakrabarti introduce very clever method to calculate the value of dv_r/dr at $r=r_s$ (e.g. Chakrabarti 1990, 1996). We extend their method to calculate the transonic solutions of the ADAFs with causal viscosity. The value of dv_r/dr at $r=r_s$ is calculated by applying L'Hopital's rule as follows. To apply L'Hopital's rule to Eq. (2) at $r=r_s$, we introduce new variables $x=r-r_s$, $y=v_r-v_{r,s}$, $z=\ell-\ell_s$ and $w=a_s-a_{s,s}$. The functions $\mathcal{D}$ and $\mathcal{N}$ are expanded near the sonic point as,

$$\mathcal{D}\ :\ \left(\frac{\partial\mathcal{D}}{\partial x}\right)_s x+\left(\frac{\partial\mathcal{D}}{\partial y}\right)_s y+\left(\frac{\partial\mathcal{D}}{\partial z}\right)_s z+\left(\frac{\partial\mathcal{D}}{\partial w}\right)_s w, \tag{16}$$

$$\mathcal{N}\ :\ \left(\frac{\partial\mathcal{N}}{\partial x}\right)_s x+\left(\frac{\partial\mathcal{N}}{\partial y}\right)_s y+\left(\frac{\partial\mathcal{N}}{\partial z}\right)_s z+\left(\frac{\partial\mathcal{N}}{\partial w}\right)_s w, \tag{17}$$

where

$$\left(\frac{\partial\mathcal{D}}{\partial x}\right)_s = \left(\frac{\partial\mathcal{D}}{\partial z}\right)_s=0,\quad \left(\frac{\partial\mathcal{D}}{\partial y}\right)_s=2,\quad \left(\frac{\partial\mathcal{D}}{\partial w}\right)_s=2\sqrt{\frac{2\gamma}{\gamma+1}},$$
$$\left(\frac{\partial\mathcal{N}}{\partial x}\right)_s = \frac{1}{(r-2)^3}-\frac{3\ell^2}{r^4}-\left(\frac{2\gamma}{\gamma+1}\right)a_s^2\frac{d^2\ln(\Omega_K/r)}{dr^2} \\ -\left(\frac{\gamma-1}{\gamma+1}\right)\frac{6\ell(\ell-j)}{r^4},$$
$$\left(\frac{\partial\mathcal{N}}{\partial y}\right)_s = 0,\quad \left(\frac{\partial\mathcal{N}}{\partial z}\right)_s=\frac{2\ell}{r^3}+\left(\frac{\gamma-1}{\gamma+1}\right)\frac{2(2\ell-j)}{r^3},$$
$$\left(\frac{\partial\mathcal{N}}{\partial w}\right)_s = \left(\frac{-4\gamma}{\gamma+1}\right)a_s\frac{d\ln(\Omega_K/r)}{dr}. \tag{18}$$

Here, $d^2\ln(\Omega_K/r)/dr^2=(5r^2-12r+12)/[2r^2(r-2)^2]$, and z and w are described as $z=(d\ell/dr)_s\,x$ and $w=(\partial a_s/\partial r)_s\,x+(\partial a_s/\partial v_r)_s\,y$ where

$$\left(\frac{\partial a_s}{\partial r}\right)_s = -\left(\frac{\gamma-1}{\gamma+1}\right)\frac{a_s}{v_r},$$
$$\left(\frac{\partial a_s}{\partial v_r}\right)_s = \left(\frac{\gamma-1}{\gamma+1}\right)\left\{a_s\frac{d\ln(\Omega_K/r)}{dr}+\frac{(\ell-j)}{a_s r^2}\left[\left(\frac{d\ell}{dr}\right)_s-\frac{2\ell}{r}\right]\right\}.$$

Finally, we obtain the quadratic equations of dv_r/dr at $r=r_s$ as, $(dv_r/dr)_s=a\pm(a^2-b)^{1/2}$ which are the solutions of the equation $(dv_r/dr)_s^2-2a(dv_r/dr)_s+b=0$ where

$$a = \frac{\left(\frac{\partial\mathcal{N}}{\partial y}\right)_s+\left(\frac{\partial\mathcal{N}}{\partial w}\right)_s\left(\frac{\partial a_s}{\partial v_r}\right)_s-\left(\frac{\partial\mathcal{D}}{\partial x}\right)_s-\left(\frac{\partial\mathcal{D}}{\partial z}\right)_s\left(\frac{\partial\ell}{\partial r}\right)_s-\left(\frac{\partial\mathcal{D}}{\partial w}\right)_s\left(\frac{\partial a_s}{\partial r}\right)_s}{2\left[\left(\frac{\partial\mathcal{D}}{\partial y}\right)_s+\left(\frac{\partial\mathcal{D}}{\partial w}\right)_s\left(\frac{\partial a_s}{\partial v_r}\right)_s\right]}, \tag{19}$$

$$b = \frac{-\left(\frac{\partial\mathcal{N}}{\partial x}\right)_s-\left(\frac{\partial\mathcal{N}}{\partial z}\right)_s\left(\frac{\partial\ell}{\partial r}\right)_s-\left(\frac{\partial\mathcal{N}}{\partial w}\right)_s\left(\frac{\partial a_s}{\partial r}\right)_s}{\left(\frac{\partial\mathcal{D}}{\partial y}\right)_s+\left(\frac{\partial\mathcal{D}}{\partial w}\right)_s\left(\frac{\partial a_s}{\partial v_r}\right)_s}. \tag{20}$$

In addition to the boundary conditions $\mathcal{D} = \mathcal{N} = 0$, when the causal viscosity prescription is adopted, the accretion flows smoothly pass the viscous radius, r_ν, where the condition $\mathcal{D}_\nu = 0$ is satisfied. In this case, the two boundary conditions at the viscous radius, r_ν, are given as $\mathcal{D}_\nu = \mathcal{N} = 0$ at $r = r_\nu$. The method presented in this study uses the boundary conditions at the sonic radius, r_s, and the viscous radius, r_ν, and do not use the boundary conditions in the outer region $r > r_\nu$ and the inner region $r_h < r < r_s$, where r_h is the radius for the event horizon.

4. Calculation Method and Sample Solutions

Based on the formula presented in the previous sections, we can calculate the global transonic solutions for ADAFs with the causal viscosity. The calculation procedures to obtain the global transonic solutions for ADAFs are as follows:

1. First, we tentatively choose some value of $a_{s,s}$ for given values of r_s and j, and calculate $v_{r,s}$, ℓ_s, $(dv_r/dr)_s$, $(d\ell/dr)_s$ and $(da_s/dr)_s$ by using Eqs. (7), (10) and (12).
2. Next, we solve the solutions in the range $r_s < r < r_\nu$. In order to do this, we solve Eqs. (7), (10) and (12) from the sonic point to the viscous point by using, e.g., the Runge-Kutta algorithm. Usually, for the initially selected value of $a_{s,s}$, the calculated solution does not pass the viscous point where two boundary conditions $\mathcal{D}_\nu = \mathcal{N}_\nu = 0$ are satisfied. In such case, we return to step 1 and again choose the different values of $a_{s,s}$ for given values of r_s and j. After repeating these procedures, we can determine the value $a_{s,s}$ which gives the solution satisfying the boundary conditions $\mathcal{D} = \mathcal{N} = 0$ at $r = r_s$ and $\mathcal{D}_\nu = \mathcal{N}_\nu = 0$ at $r = r_\nu$.
3. After solving the solutions in $r_s < r < r_\nu$, we solve Eqs. (7), (10) and (12) in the range $r_\nu < r$ by using the values of $a_{s,s}$ for given values of r_s and j by using, e.g., the Runge-Kutta algorithm.
4. Finally, we solve Eqs. (7), (10) and (12) in the range $r_h < r < r_s$ by using the values of $a_{s,s}$ for given values of r_s and j by using, e.g., the Runge-Kutta algorithm.

The third step and the fourth step can be interchanged. By this procedure, the transonic solutions are obtained for given values of r_s and j. At the third step, the physical values, such as $d\ell/dr$, at the viscous radius $r = r_\nu$ are used if necessary to obtain the solutions. Usually, the numerical calculation of the coupled equations are naturally performed through the viscous point, and the calculations in the third steps are successively done after passing through the viscous radius. However, in some cases, the numerical calculation is stopped at the viscous radius. In such case, we evaluate the physical values at the viscous point by using the L'Hopital's rule which give the value $d\ell/dr$ at the viscous radius $r = r_\nu$, and then we calculate the coupled equations for the global solution from the viscous radius to the outward direction $r > r_\nu$. In the same way as the calculation of $(dv_r/dr)_s$ as described in the last section, the value $(d\ell/dr)_{r=r_\nu}$ is calculated as $(d\ell/dr)_{r=r_\nu} = a_\nu \pm (a_\nu^2 - b_\nu)^{1/2}$ where

$$\begin{aligned}
a_\nu \;=\; & \frac{1}{2}\left\{\left(\frac{\partial \mathcal{N}_\nu}{\partial z}\right)_\nu + \left(\frac{\partial \mathcal{N}_\nu}{\partial w}\right)_\nu \left(\frac{\partial a_s}{\partial \ell}\right)_\nu - \left(\frac{\partial a_s}{\partial r}\right)_\nu - \left(\frac{\partial \mathcal{D}_\nu}{\partial x}\right)_\nu \right. \\
& \left. - \left[\left(\frac{\partial \mathcal{D}_\nu}{\partial y}\right)_\nu + \left(\frac{\partial \mathcal{D}_\nu}{\partial w}\right)_\nu \left(\frac{\partial a_s}{\partial v_r}\right)_\nu\right] \frac{\mathcal{N}_\nu}{\mathcal{D}_\nu}\right\} \\
& \Bigg/ \left[\left(\frac{\partial \mathcal{D}_\nu}{\partial z}\right)_\nu + \left(\frac{\partial \mathcal{D}_\nu}{\partial w}\right)_\nu \left(\frac{\partial a_s}{\partial \ell}\right)_\nu\right], \qquad (21) \\
b_\nu \;=\; & -\left\{\left(\frac{\partial \mathcal{N}_\nu}{\partial x}\right)_\nu + \left(\frac{\partial \mathcal{N}_\nu}{\partial w}\right)_\nu \left(\frac{\partial a_s}{\partial r}\right)_\nu\right.
\end{aligned}$$

$$+\left[\left(\frac{\partial\mathcal{N}_\nu}{\partial y}\right)_\nu+\left(\frac{\partial\mathcal{N}_\nu}{\partial w}\right)_\nu\left(\frac{\partial a_s}{\partial v_r}\right)_\nu\right]\frac{\mathcal{N}_\nu}{\mathcal{D}_\nu}\Bigg\}$$
$$\Bigg/\left[\left(\frac{\partial\mathcal{D}_\nu}{\partial z}\right)_\nu+\left(\frac{\partial\mathcal{D}_\nu}{\partial w}\right)_\nu\left(\frac{\partial a_s}{\partial \ell}\right)_\nu\right]. \tag{22}$$

Here,

$$\begin{aligned}
\left(\frac{\partial\mathcal{D}_\nu}{\partial x}\right)_\nu &= \left(\frac{\partial\mathcal{D}_\nu}{\partial z}\right)_\nu = 0, \quad \left(\frac{\partial\mathcal{D}_\nu}{\partial y}\right)_\nu = \frac{2}{\alpha^{1/2}a_s}, \quad \left(\frac{\partial\mathcal{D}_\nu}{\partial w}\right)_\nu = \frac{2}{a_s},\\
\left(\frac{\partial\mathcal{N}_\nu}{\partial x}\right)_\nu &= -\frac{\mathcal{N}_\nu}{r}-\frac{(\ell-j)}{rv_r}\frac{d(\Omega_K r)}{dr},\\
\left(\frac{\partial\mathcal{N}_\nu}{\partial y}\right)_\nu &= \frac{(\ell-j)}{rv_r}\left[\frac{v_r(\Omega_K r-2v_r)}{\alpha a_s^2}-2\right],\\
\left(\frac{\partial\mathcal{N}_\nu}{\partial z}\right)_\nu &= \frac{1}{r}\left[\frac{v_r(\Omega_K r-2v_r)}{\alpha a_s^2}+2\right],\\
\left(\frac{\partial\mathcal{N}_\nu}{\partial w}\right)_\nu &= \frac{-2(\ell-j)(\Omega_K r-2v_r)}{rv_r a_s},\\
\left(\frac{\partial a_s}{\partial r}\right) &= \left(\frac{\gamma-1}{\gamma+1}\right)\left[a_s\frac{d\ln(\Omega_K/r)}{dr}-\frac{2\ell(\ell-j)}{a_s r^3}\right],\\
\left(\frac{\partial a_s}{\partial v_r}\right) &= -\frac{a_s}{v_r}\left(\frac{\gamma-1}{\gamma+1}\right), \quad \left(\frac{\partial a_s}{\partial \ell}\right)=\left(\frac{\gamma-1}{\gamma+1}\right)\frac{(\ell-j)}{a_s r^2},
\end{aligned} \tag{23}$$

where $d(\Omega_K r)/dr = -\Omega_K(r+2)/[2(r-2)]$. The values of v_r, ℓ and a_s at the viscous radius are determined to satisfy $\mathcal{D}_\nu = \mathcal{N}_\nu = 0$.

It is noted that by using the method described above we do not use the outer boundary conditions for $r > r_\nu$ and the inner boundary conditions for $r < r_s$. If we want to the solutions for specified outer or inner boundary conditions, we should choose the values of r_s and j to satisfy the specified outer or inner boundary conditions. The correspondences between the values r_s and j and the outer solutions are investigated by Lu, Gu and Yuan (1999) for the special solutions satisfying the condition $\ell = j$ at $r = r_s$, and they provide solutions for the ADAF-thick disk solution, the ADAF-thin disk solution, and the alpha-type solution connecting the inner of outer regions. The special condition $\ell = j$ at $r = r_s$ simplify the equations described in the last section. In the case of the solutions including the cases for $\ell \neq j$ at $r = r_s$ which can be calculated by the method described above, we expect the basic picture is same as the solutions with $\ell = j$ at $r = r_s$ in Lu, Gu & Yuan (1999).

In Fig. 1, we present the sample solutions for ADAF-thick disk (*thick lines*) and ADAF-thin disk (*thin lines*) by solid lines for $\alpha = 0.1$ and $\gamma = 1.5$; the radial velocity v_r (*top-left panel*), the angular momentum (*top-right panel*), the sound speed a_s (*bottom-left panel*) and H/r (*bottom-right panel*). The filled circles and triangles denote the positions of the sonic radius, r_s, and the viscous radius, r_ν for all panels. We give the solutions in the range $r_h < r < 10^{1.5}$, and the adopted values of $\log r_s$ and j are $\log r_s = 0.7$ and $j = 2.0$. Here, we adopt the Runge-Kutta-Fehlberg algorithm for numerical calculations of the coupled differential equations of v_r, ℓ and a_s. The line of $\log|v_r|$ crosses the line of $\log[2\gamma/(\gamma+1)]^{1/2}a_s$ at the sonic radius, r_s, and the line of $\log c_\nu$ at the viscous radius, r_ν, respectively. In the outer region $r > r_\nu$, the angular momentum of the solution achieves the Keplerian value. According to the all parameter search of the ADAF solutions by Lu, Gu & Yuan (1999), the ADAF solutions are classified into the ADAF-thick disk solution, the ADAF-thin disk solution and the α-type solution connecting the inner region or the outer region. While the parameter spaces of r_s and j for the ADAF-thick solutions and the α-type solutions are

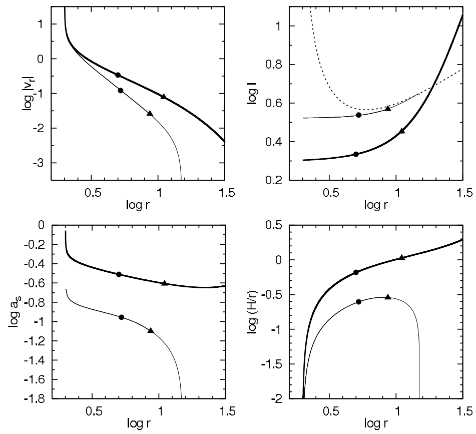

Fig. 1. The sample solutions for ADAF-thick disk (*thick lines*) and ADAF-thin disk (*thin lines*) by solid lines for $\alpha = 0.1$ and $\gamma = 1.5$; the radial velocity v_r (*top-left panel*), the angular momentum (*top-right panel*), the sound speed a_s (*bottom panel*) and H/r (*bottom-right panel*). The filled circles and triangles denote the positions of the sonic radius, r_s, and the viscous radius, r_v. In the panel of $\log \ell$, we also plot the line of the Keplerian angular momentum $\ell_K (= \Omega_K r^2)$ by a dotted line.

relatively wide, which are easily confirmed numerically, the parameter spaces for the ADAF-thin disk solutions are quite limited. By the calculation procedures presented in this paper, we can cover the all parameter spaces of the transonic solutions of ADAFs.

5. Discussion

The formula in the present study become the formula for the acausal viscous prescription if we set the diffusion timescale τ_v to be null. So, the analytic expansion around the singular points presented in this paper are also used to calculate the global solution for ADAF with the acausal viscosity. In the case of the acausal viscosity, since the boundary condition at the viscous point can not be used, the boundary condition at the other radius is required, e.g. no-torque condition at the horizon. Although the finite diffusion velocity is physically motivated, the resultant solutions have generally non-zero torque at the horizon which seems to be incorrect from the point of view of the relativity. Also, in the causal viscous prescription, the differential equation for the angular momentum become more complex than that of the acausal viscosity. So, although the finite diffusion speed is more reasonable than the infinite diffusion speed, it is noted that the calculations of the ADAF structure by using the causal viscous prescription may not be clearly better than the calculations by using the acausal viscosity.

In the fully general relativistic calculations performed by Gammie & Popham (1998) which use the causal viscosity prescription, the boundary condition at the viscous point is used when calculating the global solution. In equation (60) of Gammie & Popham (1998), although the viscous stress tensor at the horizon is finite if measured in the local rest frame, i.e. $t_{(r)(\phi)} \neq 0$, the viscous stress tensor at the horizon become null measured in the Boyer-Lindquist frame, i.e. $t^r{}_\phi = 0$, because of $\mathcal{D} = 0$ at the horizon. Since the location of the horizon is not the special location from the point of view of the local observer moving along the fluid's motion, non-zero stress in the local rest frame seems to be quite natural. The zero-torque condition arises only when we use the coordinate system

having the coordinate singularity at the horizon such as the Boyer-Lindquist coordinate and measure the viscous stress in such frames. Then, the problem of the non-zero torque or zero-torque at the horizon is the issues which require the fully relativistic treatments with the spacetime structure and the frame-transformation between the observers measuring the viscous stress. So, it may be impossible to clearly resolve the torque problem at the horizon by using the calculations with the pseudo-Newtonian potential like the present study.

6. Concluding Remarks

We present the basic equations and sample solutions for the steady-state global solutions of the ADAFs using the causal viscous prescription and the Paczyńskii-Wiita potential by the algorithm of explicit numerical integrations, such as the Runge-Kutta method. The calculation procedures to obtain these solutions are also presented. In this calculation procedure, we first solve the physical values at the sonic radius where L'Hopital's rule is used. The method presented in this paper enables us to stably solve the global solutions of ADAFs by the Runge-Kutta method, and the all parameter spaces of r_s and j for the transonic solutions of ADAFs can be covered in this method. If we set the diffusion timescale to be null, the formalism in this study includes the case of the acausal viscosity which is usually used in the past study. Since the calculation methods in this study use the analytic expansion around the singular point and the numerical integration are performed by the explicit integration, the numerical calculations become faster.

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References

Abramowicz, M. A., & Kato, S., 1989, ApJ, 336, 304
Becker, P. A., & Le, T., 2003, ApJ, 588, 408
Chakrabarti, S. K., 1990, Theory of Transonic Astrophysical Flows (Singapore: World Scientific)
Acrobat, S. K., 1996, ApJ, 464, 664
Gammie, C. F., & Popham, R., 1998, ApJ, 498, 313
Kato, S., Fukue, J., & Mineshige, S., 1998, Black-Hole Accretion Disks (Kyoto: Kyoto Univ. Press)
Lu, J.-F., Gu, W.-M., & Yuan, F., 1999, ApJ, 523, 340
Narayan, R., Kato, S., & Honma, F., 1997, ApJ, 476, 49
Paczyński, & B., Wiita, P., 1980, A& A, 88, 23
Papaloizou, J. C. B., & Szuszkiewicz, E.,1994, MNRAS, 268, 29